

Martin kernel for fractional Laplacian in narrow cones*

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Abstract

We give a power law for the homogeneity degree of the Martin kernel of the fractional Laplacian for the right circular cone when the angle of the cone tends to zero.

1 Introduction and main result

For $d \geq 2$ and $0 < \Theta < \pi$, we consider the right circular cone of angle Θ (aperture 2Θ):

$$\Gamma_\Theta = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > |x| \cos \Theta \right\}. \quad (1.1)$$

The Martin kernel of the fractional Laplacian $\Delta^{\alpha/2}$, $0 < \alpha < 2$, for Γ_Θ is the unique continuous function $M \geq 0$ on \mathbb{R}^d , such that M is smooth on Γ_Θ , $\Delta^{\alpha/2} M = 0$ on Γ_Θ , $M = 0$ on Γ_Θ^c , and $M(1, 0, \dots, 0) = 1$. It is known that M is β -homogeneous:

$$M(x) = |x|^\beta M(x/|x|), \quad x \in \mathbb{R}_0^d,$$

where $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ and $\beta = \beta(d, \alpha, \Theta) \in (0, \alpha)$. For instance, $\beta = \alpha/2$ for the half-space ($\Theta = \pi/2$). These facts are given in [1, Theorem 3.2], see also [5, Theorem 3.9]. The homogeneity degree β is crucial for precise asymptotics of nonnegative harmonic functions of $\Delta^{\alpha/2}$ in cones. In fact, the Martin, Green and heat kernels of $\Delta^{\alpha/2}$ for Γ_Θ enjoy explicit elementary estimates in terms of β [19, Lemma 3.3], [8, (23)]. By domain monotonicity of the Green function and by the boundary Harnack principle [9], the knowledge of β has important consequences for the potential theory of $\Delta^{\alpha/2}$ with Dirichlet boundary conditions in general Lipschitz domains, see, e.g. [20, Theorem 5.2]. Furthermore, β determines the critical moment $p_0 = \beta/\alpha$ of integrability of the first exit time of the isotropic α -stable Lévy process $\{X_t, t \geq 0\}$ in \mathbb{R}^d from Γ_Θ [1, Lemma 4.2], which is a long-standing motivation to study β , cf. [10], [17], [1], [12], [18]. The

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connections of $\Delta^{\alpha/2}$ to the isotropic α -stable Lévy process in \mathbb{R}^d are well-known and can be found in the references; below we focus on analytic construction of superharmonic functions of $\Delta^{\alpha/2}$ in Γ_Θ .

From [19], [1] and [17], $\beta = \beta(d, \alpha, \Theta)$ is strictly decreasing in Θ , and $\beta \rightarrow \alpha$ as $\Theta \rightarrow 0$. This contrasts with the case of the classical Laplacian, e.g. the Martin kernel for the Laplacian and planar sector with aperture $2\Theta \in (0, 2\pi)$ has homogeneity degree equal to $\pi/(2\Theta)$, which is arbitrarily large for narrow enough cones (see [10] for higher dimensions). The problem of giving a more quantitative description of β remained a puzzle for over a decade, since [1, 17]. In this work we prove a power law for $\beta(d, \alpha, \Theta)$ as $\Theta \rightarrow 0$. Namely, let $0 < \alpha < 2$,

$$\omega(\Theta) = \begin{cases} \Theta^\alpha & \text{if } 0 < \alpha < 1, \\ \Theta |\log \Theta| & \text{if } \alpha = 1, \\ \Theta & \text{if } \alpha > 1, \end{cases} \quad (1.2)$$

and

$$B_{d,\alpha} = \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{3/2}\Gamma\left(\frac{d-1+\alpha}{2}\right)} \sin\left(\frac{\pi\alpha}{2}\right) B\left(1 + \frac{\alpha}{2}, \frac{d-1}{2}\right), \quad (1.3)$$

where Γ and B are the Euler gamma and beta functions, respectively. For asymptotic results, we shall often use Landau's O notation. Here is our main theorem.

Theorem 1.1. *If $\Theta \rightarrow 0$, then $\beta(d, \alpha, \Theta) = \alpha - \Theta^{d-1+\alpha} [B_{d,\alpha} + O(\omega(\Theta))]$.*

The result is proved in Section 2.4 below. The exponent $d - 1 + \alpha$ was conjectured by Tadeusz Kulczycki in a private conversation on the methods of [17]. Here is an overview of our development. We consider the unit sphere:

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\},$$

and the spherical cap of Γ_Θ :

$$B_\Theta = \{\theta \in \mathbb{S}^{d-1} : \theta_d > \cos \Theta\}. \quad (1.4)$$

If ϕ is a function on \mathbb{S}^{d-1} , $\Phi(x) = |x|^\lambda \phi(|x|^{-1}x)$, $x \in \mathbb{R}_0^d$, and $\lambda \in (-d, \alpha)$, then we have the following decomposition:

$$\Delta^{\alpha/2} \Phi(x) = \Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(x) + R_\lambda \phi(x), \quad x \in \mathbb{S}^{d-1}.$$

The spherical part $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}$ is an integro-differential operator on \mathbb{S}^{d-1} akin to the fractional Laplacian in dimension $d - 1$. The radial part R_λ is an integral operator on \mathbb{S}^{d-1} whose nonnegative kernel increases in $\lambda \in ((\alpha - d)/2, \alpha)$, in fact explodes as $\lambda \rightarrow \alpha$. We have $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} M = -R_\beta M$ on B_Θ . Heuristically, β is a generalized eigenvalue of $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}$ with Dirichlet conditions, relative to the family R_λ . In the classical case $\alpha = 2$, the operator R_γ reduces to multiplication by γ , which leads to a genuine eigenproblem on the sphere (see e.g. [11]). To estimate β we define a suitable spherical profile function ϕ on \mathbb{S}^{d-1} supported on B_Θ , extend it to be λ -homogeneous on \mathbb{R}^d and choose λ so that the extension is either superharmonic or subharmonic for $\Delta^{\alpha/2}$. This yields lower

and upper bounds for M and β by means of the maximum principle for $\Delta^{\alpha/2}$. Namely, $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi$ is provided by a judicious choice of ϕ , and we control $\alpha - \beta$ by proving uniform estimates for the kernel of R_λ .

Two candidates offer themselves to construct ϕ : the principal eigenfunction of the fractional Laplacian for the ball in dimension $d - 1$ and the expected exit time of the isotropic α -stable Lévy process from the ball. Surprisingly, it is the latter choice that allows us to handle the super- and subharmonicity of Φ for $\Delta^{\alpha/2}$ up to the boundary of Γ_Θ . The expected exit time of the ball has the additional advantage of being explicit, allowing us to construct explicit barriers (i.e. superharmonic functions vanishing at the boundary) for narrow cones. We remark that the principal eigenfunction and eigenvalue of the ball for $\Delta^{\alpha/2}$ are not known (see [13] for bounds and references), therefore the above expression for $B_{d,\alpha}$ is a remarkable serendipity. We note in passing the Martin kernel of Γ_Θ with the pole at the origin is $|x|^{\alpha-d} M(x/|x|^2)$, $x \in \mathbb{R}_0^d$ [1], hence its singularity at the origin is roughly $|x|^{-d+B_{d,\alpha}\Theta^{d-1+\alpha}}$ for small Θ . This exemplifies some of the extreme behaviour of nonnegative α -harmonic functions at the boundary of (narrow) Lipschitz open sets.

The structure of the paper is as follows. The main line of arguments is presented in Section 2, where we give preliminaries, detail the above decompositions of $\Delta^{\alpha/2}$ and state precise asymptotic results for the kernels of the spherical and radial operators. As mentioned, the spherical profile ϕ is constructed from the expected exit time of the ball for the isotropic α -stable Lévy process in \mathbb{R}^{d-1} . We also use a suitable variant of Kelvin transform to define ϕ . We then estimate $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi + R_\lambda \phi$. The proof of Theorem 1.1 is given at the end of Section 2. In Section 3 we collect the more technical proofs from Section 2 and some auxiliary results. We also prove the following result for the classical Laplacian in the complement of a plane slit by a cone, using Theorem 1.1 and the connection of Δ and $\Delta^{1/2}$ in dimensions d and $d - 1$, respectively.

Corollary 1.2. *Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0 \text{ or } x_2 > \sqrt{x_1^2 + x_2^2} \cos \Theta\}$. The homogeneity exponent of Martin kernel of Δ and V is $1 - \Theta^2/4 + O(\Theta^3 \log \Theta)$ as $\Theta \rightarrow 0^+$.*

Our work leads to interesting new problems. It is worthwhile to study the above generalized eigenproblem of $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}$ in the setting of $L^2(\mathbb{S}^{d-1}, \sigma)$, as opposed to the present pointwise setting. Some results in this direction are given in [1]. It would be very interesting to understand the generalized higher-order eigenfunctions of $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}$ with respect to R_λ and, ultimately, oscillating harmonic functions of $\Delta^{\alpha/2}$. We note that nonnegative harmonic functions of $\Delta^{\alpha/2}$ have been completely described in [9], but oscillating harmonic functions of the operator are hardly understood. Similar problems are relevant for more general nonlocal Lévy-type operators, of which $\Delta^{\alpha/2}$ is but a prominent example. For instance, the homogeneity of the Martin kernel of cones for more general stable Lévy processes should now be available by using the methods of [1] and the boundary Harnack principle recently proved in [3].

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2 Homogeneous superharmonic functions

2.1 Decomposition of the fractional Laplacian

Below we let $d \geq 2$ and $0 < \alpha < 2$, unless explicitly stated otherwise. (The reader may consult [1] for $d = 1$ and [10] for $\alpha = 2$.) All sets, functions and measures on \mathbb{R}^d considered below are assumed Borel. By dx we denote the Lebesgue measure on \mathbb{R}^d or \mathbb{R}^{d-1} , depending on context. We let σ be the $(d-1)$ -dimensional Hausdorff measure (surface measure), so normalized that

$$\omega_d := \sigma(\mathbb{S}^{d-1}) = 2\pi^{d/2}/\Gamma(d/2).$$

Let $0 < \Theta < \pi$ and $\mathbf{1} = (0, 0, \dots, 0, 1)$. We have

$$\Gamma_\Theta = \left\{ x \in \mathbb{R}^d : x \cdot \mathbf{1} > |x| \cos \Theta \right\} \quad \text{and} \quad B_\Theta = \left\{ \theta \in \mathbb{S}^{d-1} : \theta \cdot \mathbf{1} > \cos \Theta \right\}.$$

Let

$$\mathcal{A}_{d,\alpha} = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma\left(\frac{d+\alpha}{2}\right) / \Gamma\left(1 - \frac{\alpha}{2}\right).$$

For a real-valued function Φ on \mathbb{R}^d , twice continuously differentiable, i.e. C^2 near some point $x \in \mathbb{R}^d$ and such that

$$\int_{\mathbb{R}^d} |\Phi(y)| (1 + |y|)^{-d-\alpha} dy < \infty, \quad (2.1)$$

we define

$$\Delta^{\alpha/2} \Phi(x) = \lim_{\epsilon \downarrow 0} \mathcal{A}_{d,\alpha} \int_{|y-x| > \epsilon} [\Phi(y) - \Phi(x)] |y-x|^{-d-\alpha} dy, \quad (2.2)$$

the fractional Laplacian of Φ at x , and we say Φ is α -harmonic, i.e. harmonic for $\Delta^{\alpha/2}$, on open $D \subset \mathbb{R}^d$ if $\Delta^{\alpha/2} \Phi(x) = 0$ for $x \in D$ (see [5, 6, 7] for broader discussion). If $x \notin \text{supp } \Phi$, then

$$\Delta^{\alpha/2} \Phi(x) = \mathcal{A}_{d,\alpha} \int_{\mathbb{R}^d} \Phi(y) |y-x|^{-d-\alpha} dy. \quad (2.3)$$

If $r > 0$ and $\Phi_r(x) = \Phi(rx)$, then the following scaling property holds:

$$\Delta^{\alpha/2} \Phi_r(x) = r^\alpha \Delta^{\alpha/2} \Phi(rx), \quad x \in \mathbb{R}^d. \quad (2.4)$$

In this respect, $\Delta^{\alpha/2}$ behaves like differentiation of order α .

Let $C_{d,\alpha} = \Gamma\left(\frac{d}{2}\right) / [2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(1 + \frac{\alpha}{2}\right)]$ and define

$$S(x) = C_{d,\alpha} (1 - |x|^2)^{\alpha/2}, \quad \text{if } |x| \leq 1, \quad (2.5)$$

and $S(x) = 0$ if $|x| > 1$. We then have

$$\Delta^{\alpha/2} S(x) = -1, \quad |x| < 1. \quad (2.6)$$

Indeed, $S(x)$ is identified in [14] with the expected time of the first exit from the unit ball for the isotropic α -stable Lévy process starting at $x \in \mathbb{R}^d$, from which (2.6) follows, cf. [6, Lemma 5.3] and [5, Lemma 3.8]. An alternative approach to (2.5) and further

probabilistic connections may be found in [6, (5.4)] and [14]. A direct purely analytic proof of (2.6) is given in [13], cf. Table 3 ibid.

For $\lambda \in (-d, \alpha)$ we consider, after [1, Section 5], the following kernel

$$u_\lambda(t) = \int_0^\infty r^{d+\lambda-1} (r^2 - 2rt + 1)^{-(d+\alpha)/2} dr, \quad (2.7)$$

$$= \int_0^1 r^{-1} (r^{d+\lambda} + r^{\alpha-\lambda}) (r^2 - 2rt + 1)^{-(d+\alpha)/2} dr, \quad -1 \leq t < 1 \quad (2.8)$$

(we drop d and α from the notation). Since $r^2 - 2rt + 1 = (r-t)^2 + 1 - t^2$, we see from (2.7) that $u_\lambda(t) < \infty$ if $-1 \leq t < 1$ and $u_\lambda(1) = \infty$. By (2.8), $[-1, 1) \ni t \mapsto u_\lambda(t)$ is strictly increasing. Furthermore, for every $t \in [-1, 1)$, $\lambda \mapsto u_\lambda(t)$ is strictly increasing on $[(\alpha - d)/2, \alpha)$ and $\lim_{\lambda \rightarrow \alpha} u_\lambda(t) = \infty$ (see [1, Lemma 5.2] or below).

The following result is given in [1, (37)].

Lemma 2.1. *If Φ is real-valued, homogeneous of degree $\lambda \in (-d, \alpha)$ on \mathbb{R}^d , bounded on \mathbb{S}^{d-1} and C^2 near a point $\eta \in \mathbb{S}^{d-1}$, and if ϕ is the restriction of Φ to \mathbb{S}^{d-1} , then*

$$\Delta^{\alpha/2} \Phi(\eta) = \Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(\eta) + R_\lambda \phi(\eta),$$

where the spherical fractional Laplacian is

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(\eta) = \mathcal{A}_{d,\alpha} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{S}^{d-1} \setminus \{\theta \cdot \eta > \cos \epsilon\}} [\phi(\theta) - \phi(\eta)] u_0(\theta \cdot \eta) \sigma(d\theta), \quad (2.9)$$

and the radial operator of order λ is

$$R_\lambda \phi(\eta) = \mathcal{A}_{d,\alpha} \int_{\mathbb{S}^{d-1}} \phi(\theta) [u_\lambda(\theta \cdot \eta) - u_0(\theta \cdot \eta)] \sigma(d\theta). \quad (2.10)$$

We may consider the equation $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(\eta) = \Delta^{\alpha/2} \Phi(\eta)$, $\eta \in \mathbb{S}^{d-1}$, as the definition of $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi$, if Φ is the 0-homogeneous extension of ϕ to \mathbb{R}_0^d .

For two nonnegative functions f and g on a set D , we say that f is comparable to g and write $f \asymp g$, or $f(x) \asymp g(x)$ for $x \in D$, if constant C exists such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x), \quad x \in D.$$

Here constant means a positive number independent of x . If not specified otherwise constants depend only on d and α . If we write $C = C(a, \dots, z)$, then we mean that C depends only on a, \dots, z . The actual value of a constant may change from line to line. For instance, $M(\theta) \asymp \text{dist}(\theta, \Gamma^c)^{\alpha/2}$, $\theta \in \mathbb{S}^{d-1}$, which is related to the fact that Γ is smooth except at the vertex, and so

$$M(x) \asymp |x|^{\beta-\alpha/2} \text{dist}(x, \Gamma^c)^{\alpha/2}, \quad x \in \mathbb{R}_0^d,$$

see [19, Lemma 3.3]. We say that ϕ defined on \mathbb{S}^{d-1} is C^2 on a part of the sphere if its 0-homogeneous extension to \mathbb{R}^d is C^2 in a neighborhood of this part of the sphere.

The following lemma helps estimate the homogeneity degree β of M .

Lemma 2.2. *Let $\phi \in C(\mathbb{S}^{d-1}) \cap C^2(B_\Theta)$ and $\phi(\theta) \asymp \text{dist}^{\alpha/2}(\theta, \Gamma_\Theta^c)$ for $\theta \in \mathbb{S}^{d-1}$.*

If $\lambda \in (0, \alpha)$ is such that $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi + R_\lambda \phi \geq 0$ on B_Θ , then $\lambda \geq \beta(\alpha, \Theta)$.

If $\lambda \in (0, \alpha)$ is such that $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi + R_\lambda \phi \leq 0$ on B_Θ , then $\lambda \leq \beta(\alpha, \Theta)$.

Proof. Suppose that $0 < \lambda < \beta(\alpha, \Theta)$. Let $\Phi(0) = 0$ and $\Phi(x) = |x|^\lambda \phi(x/|x|)$ for $x \in \mathbb{R}_0^d$. Note that $\Delta^{\alpha/2} \Phi(x) = |x|^{\lambda-\alpha} \Delta^{\alpha/2} \Phi(x/|x|)$ for $x \in \Gamma_\Theta$. The function $h = \Phi - M$ is continuous on \mathbb{R}^d , C^2 on Γ_Θ and vanishes on Γ_Θ^c . Since Φ and M are comparable on \mathbb{S}^{d-1} , $h(x) < 0$ for large enough $x \in \Gamma_\Theta$ and $h(x) > 0$ for small enough $x \in \Gamma_\Theta$. Therefore h has a global positive maximum at some $x_0 \in \Gamma_\Theta$. Considering the integration on Γ_Θ^c in (2.2) and (2.4) we obtain

$$\begin{aligned} 0 &> \Delta^{\alpha/2} h(x_0) = \Delta^{\alpha/2} \Phi(x_0) - \Delta^{\alpha/2} M(x_0) = \Delta^{\alpha/2} \Phi(x_0) \\ &= |x_0|^{\lambda-\alpha} \left[\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(x_0/|x_0|) + R_\lambda \phi(x_0/|x_0|) \right]. \end{aligned}$$

This yields the first assertion, and the second one is proved similarly. \square

2.2 Inversion

We shall construct functions ϕ satisfying the assumptions of Lemma 2.2 by using an appropriate Kelvin transform. The inversion T with respect to the unit sphere \mathbb{S}^{d-1} is

$$Tx = \frac{1}{|x|^2} x, \quad x \in \mathbb{R}_0^d.$$

Note that T^2 is the identity of \mathbb{R}_0^d , and $T(\frac{1}{2}\mathbf{1}) = 2\mathbf{1}$. The inversion preserves angles and the class of all straight lines and circles on \mathbb{R}_0^d , because

$$|Tx - Ty| = \frac{|x - y|}{|x||y|}, \quad x, y \in \mathbb{R}_0^d. \quad (2.11)$$

The Kelvin transform K appropriate for $\Delta^{\alpha/2}$ is defined, for functions v on \mathbb{R}_0^d , as follows

$$(Kv)(y) = |y|^{\alpha-d} v(Ty), \quad y \in \mathbb{R}_0^d.$$

Thus, $K^2 v = v$. We have

$$\Delta^{\alpha/2} [Kv](y) = |y|^{-\alpha-d} \Delta^{\alpha/2} v(Ty), \quad y \in \mathbb{R}_0^d. \quad (2.12)$$

The formula is given in [4, p. 112] as a consequence of a transformation rule for Green potentials of $\Delta^{\alpha/2}$, cf. (71) and (72) *ibid*. In particular, if v is α -harmonic on open set $D \subset \mathbb{R}_0^d$, then Kv is α -harmonic on TD . We define the Riesz kernel

$$h(y) = |y|^{\alpha-d}, \quad (2.13)$$

and recall that h is α -harmonic on \mathbb{R}_0^d . In fact, $h = K1$.

For $y = (y_1, \dots, y_{d-1}, y_d) \in \mathbb{R}^d$ we let $\tilde{y} = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$, so that $y = (\tilde{y}, y_d)$. We consider the shifted cone

$$V_\Theta = \Gamma_\Theta + \frac{1}{2}\mathbf{1},$$

and the sphere of radius $1/2$ centered at $\frac{1}{2}\mathbf{1}$, which we denote by

$$\mathcal{S} = \left\{ z \in \mathbb{R}^d : |z - \frac{1}{2}\mathbf{1}| = \frac{1}{2} \right\},$$

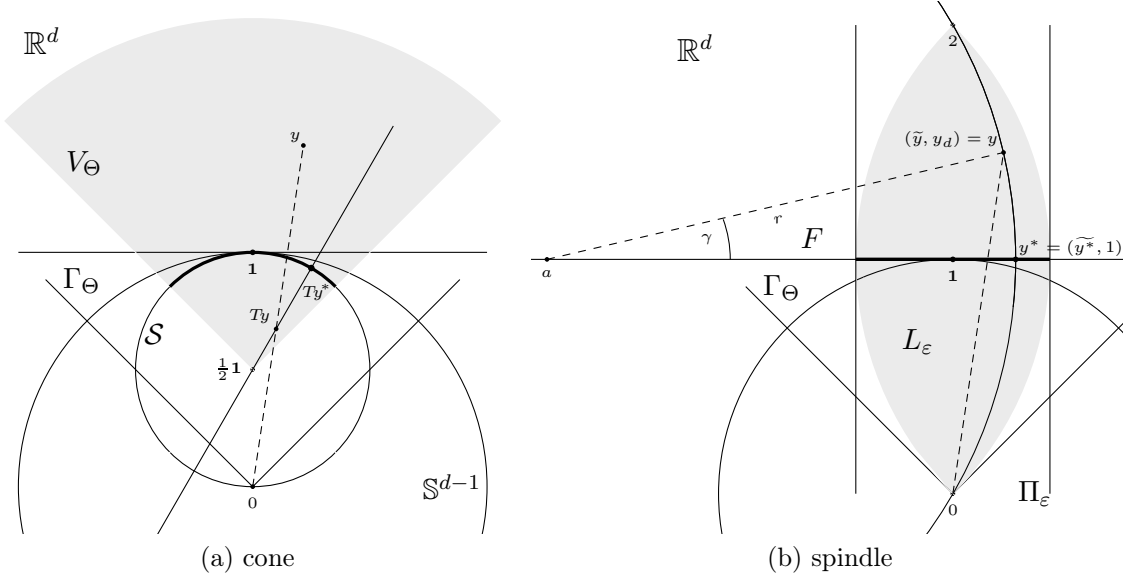


Figure 1: The shifted cone V_Θ and its inversion L_ϵ

see Figure 1a. In particular, as shown on Figure 1b, the inversion of \mathcal{S} is flat:

$$F := TS = \{y = (\tilde{y}, y_d) : y_d = 1\}.$$

Let ϵ be the radius of $T(\mathcal{S} \cap V_\Theta)$, that is

$$\epsilon = \tan \frac{\Theta}{2}. \quad (2.14)$$

Since Theorem 1.1 is asymptotic, in what follows we may and do assume that $\epsilon < 1/20$. We consider the cylinder

$$\Pi_\epsilon = \{y = (\tilde{y}, y_d) : |\tilde{y}| < \epsilon\},$$

and the spindle-shaped image of V_Θ by T :

$$L_\epsilon = TV_\Theta,$$

see Figure 1b, which is tangent to the boundary of Π_ϵ , cf. (2.14). For $y \in L_\epsilon$, we denote by y^* the intersection point of F , Π_ϵ and the circle (or line) passing through 0, $2\mathbf{1}$ and y . Thus, y^* is a curvilinear projection of y on F . Equivalently, Ty^* is the intersection point of \mathcal{S} and the ray from $\frac{1}{2}\mathbf{1}$ through Ty . Note that $y_d^* = 1$. We claim that

$$||\tilde{y}| - |\tilde{y}^*|| \leq \frac{1}{2}|y - y^*|^2, \quad y \in \mathbb{R}_0^d. \quad (2.15)$$

Indeed, if $\tilde{y} = 0$, then $\tilde{y}^* = 0$, and (2.15) is trivial. Else 0, $2\mathbf{1}$ and y are not collinear, and we let a denote the center of the circle through these points, r its radius and γ the angle between the lines ay and ay^* . We then observe that $r \geq 1$, $|y - y^*| = 2r \sin(\gamma/2)$, and $||\tilde{y}| - |\tilde{y}^*|| = r - r \cos \gamma = 2r \sin^2(\gamma/2) = |y - y^*|^2/(2r) \leq |y - y^*|^2/2$, as claimed, cf. Figure 1b. We let

$$s_\epsilon(y) = C_{d-1,\alpha}(\epsilon^2 - |\tilde{y}|^2)_+^{\alpha/2}, \quad y \in \mathbb{R}^d.$$

We have

$$\Delta^{\alpha/2} s_\varepsilon(y) = -1, \quad y \in \Pi_\varepsilon. \quad (2.16)$$

Indeed, \widetilde{X}_t is the isotropic α -stable Lévy process in \mathbb{R}^{d-1} , and the first exit time of \widetilde{X}_t from \widetilde{B} is s_ε , the same as the expected exit time of X_t from Π_1 . This yields (2.16), cf. [6, Lemma 5.3 and p. 319] and [5, Lemma 3.8]. For $x, y \in \mathbb{R}^d$ we have

$$|s_\varepsilon(y) - s_\varepsilon(x)| \leq 2C_{d-1,\alpha} \varepsilon^\alpha ||\widetilde{y}/\varepsilon| - |\widetilde{x}/\varepsilon||^{\alpha/2} \leq 2C_{d-1,\alpha} \varepsilon^{\alpha/2} |y - x|^{\alpha/2}. \quad (2.17)$$

We also define

$$s_\varepsilon^*(y) = \begin{cases} C_{d-1,\alpha} (\varepsilon^2 - |\widetilde{y}^*|^2)_+^{\alpha/2}, & \text{if } y \in L_\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

2.3 Main estimates

In this section we present a chain of estimates. As we shall see in Section 2.4, they lead to functions ϕ satisfying the assumptions of Lemma 2.2, and so yield Theorem 1.1.

Recall that ω is defined in (1.2) and the Riesz kernel h is defined in (2.13).

Lemma 2.3. *For $x \in \Pi_\varepsilon \cap F$ we have $\Delta^{\alpha/2}(h s_\varepsilon)(x) = -h(x) + O(\omega(\varepsilon))$.*

The proof of Lemma 2.3 is given in Section 3.

Lemma 2.4. *For $x \in \Pi_\varepsilon \cap F$ we have $\Delta^{\alpha/2}[h(s_\varepsilon^* - s_\varepsilon)](x) = O(\varepsilon^\alpha)$.*

The proof of Lemma 2.4 is given in Section 3. We define

$$u = K(h s_\varepsilon^*), \quad (2.18)$$

or

$$u(y) = s_\varepsilon^*(Ty). \quad (2.19)$$

Note that u is supported on V_Θ and constant on rays from $\frac{1}{2}\mathbf{1}$, since $s_\varepsilon^*(y) = s_\varepsilon^*(y^*)$ for all $y \in \mathbb{R}_0^d$. If $x \in \mathcal{S} \cap V_\Theta$, then $|x - \mathbf{1}/2| = 1/2$ and $x^* = x$. Note that $|Tx| = 1/|x|$ and, by (2.11), $|Tx - \mathbf{1}| = |Tx - T\mathbf{1}| = |x - \mathbf{1}|/|x|$. This simplifies (2.18) as follows

$$u(x) = C_{d-1,\alpha} (\varepsilon^2 - |Tx - \mathbf{1}|^2)_+^{\alpha/2} = C_{d-1,\alpha} \left(\varepsilon^2 - \frac{|x - \mathbf{1}|^2}{|x|^2} \right)_+^{\alpha/2}, \quad x \in \mathbb{R}_0^d.$$

For $\theta \in \mathbb{S}^{d-1}$ we define the profile function,

$$\phi(\theta) := 2^\alpha u(\theta/2 + \mathbf{1}/2) = 2^\alpha C_{d-1,\alpha} \left(\varepsilon^2 - \frac{|\mathbf{1} - \theta|^2}{|\mathbf{1} + \theta|^2} \right)_+^{\alpha/2}. \quad (2.20)$$

Proposition 2.5. *For $\theta \in \mathcal{S} \cap V_\Theta$ we have*

$$\Delta^{\alpha/2} u(\theta) = -|\theta|^{-2\alpha} + O(\omega(\varepsilon)).$$

Proof. Note that $\varepsilon^\alpha = O(\omega(\varepsilon))$. By Lemma 2.3, Lemma 2.4, (2.12) and (2.15) we get the result. \square

Corollary 2.6. *We have $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}\phi(\eta) = -1 + O(\omega(\Theta))$ for $\eta \in B_\Theta$.*

Proof. By Proposition 2.5 for $\theta \in \mathcal{S} \cap V_\Theta$, we have

$$\Delta^{\alpha/2}u(x) = -1 + O(\omega(\varepsilon)).$$

We consider $x \mapsto 2^\alpha u(x/2 + \mathbf{1}/2)$. The function is homogeneous of order 0. By (2.4) $\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}\phi(\theta) = (\Delta^{\alpha/2}u)(\theta/2 + \mathbf{1}/2) = -1 + O(\omega(\Theta))$. \square

To verify the assumptions of Lemma 2.2 we need to estimate $R_\lambda\phi$, cf. (2.10).

Lemma 2.7. *If $0 < \lambda < \alpha < \lambda + 1$ and $0 < \delta < 1$, then for $c = c(d)$ and $C = C(d)$,*

$$\frac{1}{\alpha - \lambda} - c \leq u_\lambda(t) - u_0(t) \leq \frac{1}{\alpha - \lambda} + \frac{C}{(\alpha - \lambda)^{1-\delta}} + \frac{C[1 \vee (1-t)^{-(d+\alpha-3)/2}]}{(\alpha - \lambda)^\delta}, \quad t \in [-1, 1].$$

The proof of Lemma 2.7 is given in Section 3.

Lemma 2.8. *Let $\tilde{C}_{d,\alpha} = \frac{1}{2}\omega_{d-1}C_{d-1,\alpha}B\left(1 + \frac{\alpha}{2}, \frac{d-1}{2}\right)$. We have*

$$\int_{\mathbb{S}^{d-1}} \phi(\theta)\sigma(d\theta) = \tilde{C}_{d,\alpha}\Theta^{\alpha+d-1} [1 + O(\Theta^2)], \quad \text{as } \Theta \rightarrow 0^+.$$

The proof of Lemma 2.8 is given in Section 3.

Lemma 2.9. *Let $B_{d,\alpha} = \mathcal{A}_{d,\alpha}\tilde{C}_{d,\alpha}$, as in (1.3). For all $\eta \in B_\Theta$ we have*

$$R_\lambda\phi(\eta) \geq B_{d,\alpha}\Theta^{d-1+\alpha} [1 + O(\Theta^2)] \frac{1 - c(\alpha - \lambda)}{\alpha - \lambda}, \quad (2.21)$$

and, under the condition $0 < \delta < 1$,

$$R_\lambda\phi(\eta) \leq B_{d,\alpha}\Theta^{d-1+\alpha} [1 + O(\Theta^2)] \frac{1 + C(\alpha - \lambda)^\delta}{\alpha - \lambda} + C \frac{\Theta\omega(\Theta)}{(\alpha - \lambda)^\delta}. \quad (2.22)$$

The proof of Lemma 2.9 is given in Section 3.

2.4 Proof of Theorem 1.1

Let

$$\lambda = \alpha - B_{d,\alpha}\Theta^{d-1+\alpha}[1 + \kappa\omega(\Theta)],$$

where $\kappa > 0$ shall be defined later. We let $\delta = (d + \alpha - 1)^{-1}$ in (2.22) and obtain

$$R_\lambda\phi(\eta) \leq (1 + c_1\Theta^2) \frac{1 + c_2\Theta}{1 + \kappa\omega(\Theta)} + c_3\omega(\Theta), \quad \eta \in B_\Theta.$$

If $0 \leq a \leq b < 1$, then $(1 + a)/(1 + b) \leq 1 - (b - a)/2$. If $\kappa \geq c_2$ and $\kappa\omega(\Theta) < 1$, then

$$R_\lambda\phi(\eta) \leq 1 + (-\kappa/2 + c_2/2 + c_1 + c_3)\omega(\Theta), \quad \eta \in B_\Theta.$$

By Corollary 2.6,

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2}\phi(\eta) \leq -1 + c_4\omega(\Theta), \quad \eta \in B_\Theta.$$

Accordingly, we stipulate $\kappa \geq 2c_1 + c_2 + 2c_3 + 2c_4$. For $\omega(\Theta) < 1/\kappa$ we then have

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(\eta) + R_\lambda \phi(\eta) \leq 0, \quad \eta \in B_\Theta,$$

and [Lemma 2.2](#) yields

$$\beta(\alpha, \Theta) \geq \alpha - B_{d,\alpha} \Theta^{d-1+\alpha} [1 + \kappa \omega(\Theta)].$$

To obtain an opposite bound, we put

$$\lambda = \alpha - B_{d,\alpha} \frac{\Theta^{d-1+\alpha}}{1 + \iota \omega(\Theta)},$$

and we shall define $\iota > 0$ momentarily. By [\(2.21\)](#),

$$\begin{aligned} R_\lambda \phi(\eta) &\geq \frac{1 - c(\alpha - \lambda)}{\alpha - \lambda} B_{d,\alpha} \Theta^{d-1+\alpha} [1 + O(\Theta^2)] \\ &\geq 1 - c' \Theta^{d-1+\alpha} + \iota \omega(\Theta) - c'' \Theta^2. \end{aligned}$$

Recall that $d \geq 2$ and $\Theta \leq \omega(\Theta)$. Taking $\iota \geq c' + c''$, by [Corollary 2.6](#) we get

$$\Delta_{\mathbb{S}^{d-1}}^{\alpha/2} \phi(\eta) + R_\lambda \phi(\eta) \geq 0,$$

and [Lemma 2.2](#) yields

$$\beta(\alpha, \Theta) \leq \alpha - B_{d,\alpha} \Theta^{d-1+\alpha} [1 - \iota \omega(\Theta)/2],$$

provided $\omega(\Theta) < 1/(2\iota)$. This ends the proof of [Theorem 1.1](#).

3 Technical details

We now give proofs of the more technical lemmas from [Section 2](#), and further results.

3.1 Proof of [Lemma 2.3](#)

For $x \in \Pi_\varepsilon \cap F$ we have

$$\begin{aligned} \Delta^{\alpha/2}(h s_\varepsilon)(x) &= -h(x) + \left[\Delta^{\alpha/2}(h s_\varepsilon)(x) - h(x) \Delta^{\alpha/2} s_\varepsilon(x) - s_\varepsilon(x) \Delta^{\alpha/2} h(x) \right] \\ &= -h(x) + \lim_{\epsilon \rightarrow 0} \mathcal{A}_{d,\alpha} \int_{|x-y| \geq \epsilon} \frac{[s_\varepsilon(y) - s_\varepsilon(x)][h(y) - h(x)]}{|x-y|^{d+\alpha}} dy. \end{aligned} \quad (3.1)$$

To analyze the integral in [\(3.1\)](#), we define the following sets

$$G = \{y \in \mathbb{R}^d : |\tilde{y}| < 1/2, |y_d| \leq 1/2\},$$

$$H = \{y \in \mathbb{R}^d : |\tilde{y}| < 1/2, |y_d - 1| \leq 1/2\}.$$

Recall that $s_\varepsilon(y) \leq c_{d-1,\alpha} \varepsilon^\alpha$, $y \in \mathbb{R}^d$. Observe that $h(x) \leq 1$. For $y \in (G \cup H)^c$ we have $|y| > 1/2$ and $|x - y| > c|y|$, hence

$$\int_{(G \cup H)^c} \frac{|s_\varepsilon(x) - s_\varepsilon(y)| |h(x) - h(y)|}{|x - y|^{d+\alpha}} dy \leq c \varepsilon^\alpha \int_{(G \cup H)^c} (|y|^{\alpha-d} + 1) |y|^{-d-\alpha} dy = c \varepsilon^\alpha.$$

On G we have $|x - y| \geq 1/2$, and

$$\int_G |s_\varepsilon(x) - s_\varepsilon(y)| |h(x) - h(y)| |x - y|^{-d-\alpha} dy \leq C_{d-1,\alpha} \varepsilon^\alpha 2^{d+\alpha} \int_G h(y) dy \leq c\varepsilon^\alpha.$$

Note that $H \subset B(x, 1)$. Let $\delta = \varepsilon - |\tilde{x}|$, the distance from x to Π_ε^c . Then $s_\varepsilon(x) \leq c\delta^{\alpha/2} \varepsilon^{\alpha/2}$, $x \in \Pi_\varepsilon \cap F$. We split the integral on H into $H_1 = H \setminus \Pi_\varepsilon$ and $H_2 = H \cap \Pi_\varepsilon$. Since h is Lipschitz and $s_\varepsilon(y) = 0$ on H_1 , we get

$$\begin{aligned} I_1 &:= \int_{H_1} \frac{|s_\varepsilon(x) - s_\varepsilon(y)| |h(x) - h(y)|}{|x - y|^{d+\alpha}} dy \leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} \int_{H_1} \frac{1}{|x - y|^{d-1+\alpha}} dy \\ &\leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} \int_{B(x,1) \setminus B(x,\delta)} \frac{1}{|x - y|^{d-1+\alpha}} dy \end{aligned}$$

If $\alpha < 1$, then the last integral is bounded by

$$\int_{B(x,1)} \frac{1}{|x - y|^{d-1+\alpha}} dy \leq c$$

and, $\delta \leq \varepsilon$, implies $I_1 \leq c\varepsilon^\alpha$. If $\alpha > 1$, then

$$\begin{aligned} I_1 &\leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} \int_{B(x,\delta)^c} |x - y|^{-d-\alpha+1} dy \\ &\leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} \delta^{1-\alpha} \\ &= c\delta^{1-\alpha/2} \varepsilon^{\alpha/2} \\ &\leq c\varepsilon^{1-\alpha/2} \varepsilon^{\alpha/2} = c\varepsilon. \end{aligned}$$

Finally, for $\alpha = 1$ we have

$$\begin{aligned} I_1 &\leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} \int_{B(x,1) \setminus B(x,\delta)} |x - y|^{-d-\alpha+1} dy \\ &\leq c\delta^{\alpha/2} \varepsilon^{\alpha/2} |\ln \delta|. \end{aligned}$$

But $\sqrt{\delta} |\ln \delta|$ is increasing on $0 < \delta < e^{-2}$, while we have $\delta \leq \varepsilon < 1/20$. Hence we can replace δ with ε in the last line. Summarizing, for any $\alpha \in (0, 2)$ we have $I_1 \leq c\omega(\varepsilon)$.

Recall that by (2.17), for any x and y , we have $|s_\varepsilon(x) - s_\varepsilon(y)| \leq c\varepsilon^{\alpha/2} |x - y|^{\alpha/2}$. Hence,

$$\begin{aligned} \int_{H_2} \frac{|s_\varepsilon(x) - s_\varepsilon(y)| |h(x) - h(y)|}{|x - y|^{d+\alpha}} dy &\leq \int_{B(x,1)} \frac{|s_\varepsilon(x) - s_\varepsilon(y)| |h(x) - h(y)|}{|x - y|^{d+\alpha}} dy \\ &\leq c\varepsilon^{\alpha/2} \int_{B(x,1)} |x - y|^{-d+(1-\alpha/2)} dy \\ &\leq c\varepsilon^{\alpha/2} \delta^{1-\alpha/2} \leq c\varepsilon. \end{aligned}$$

But $\varepsilon^\alpha \leq \omega(\varepsilon)$ and $\varepsilon \leq \omega(\varepsilon)$, ending the proof of Lemma 2.3.

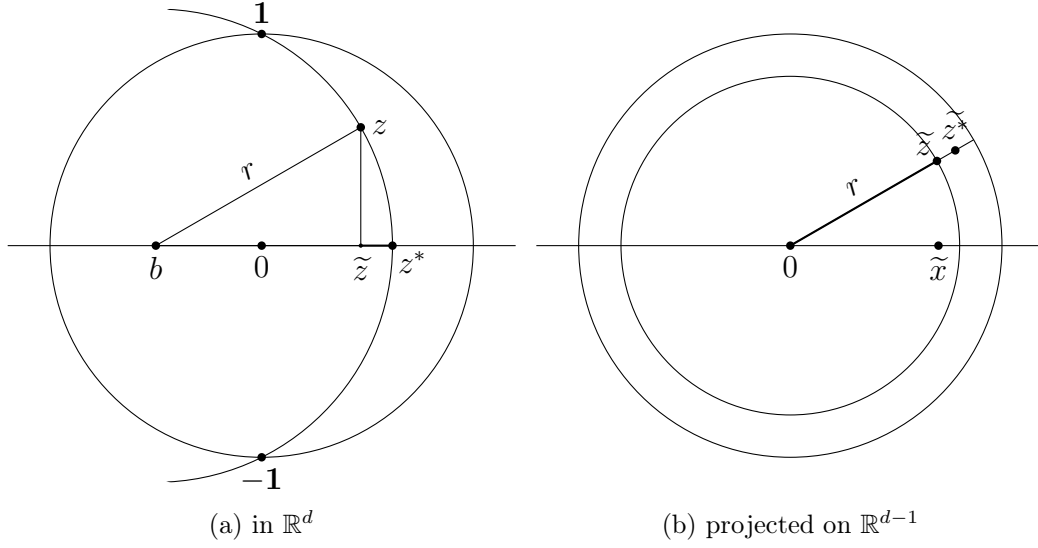


Figure 2: Coordinates z_d, z^* .

3.2 Proof of Lemma 2.4

For $x \in \Pi_\varepsilon \cap F$ we have $x^* = x$ and $s_\varepsilon^*(x) - s_\varepsilon(x) = 0$. Furthermore, for $y \in \Pi_\varepsilon$ we have $s_\varepsilon^*(y) \leq s_\varepsilon(y)$. Therefore,

$$\begin{aligned} \Delta^{\alpha/2}[h(s_\varepsilon^* - s_\varepsilon)](x) &= \lim_{\varepsilon \rightarrow 0} \mathcal{A}_{d,\alpha} \int_{\Pi_\varepsilon \cap \{|x-y| \geq \varepsilon\}} h(y)[s_\varepsilon(y) - s_\varepsilon^*(y)]|x-y|^{-d-\alpha} dy \\ &= \mathcal{A}_{d,\alpha} \int_{\Pi_\varepsilon} h(y)[s_\varepsilon(y) - s_\varepsilon^*(y)]|x-y|^{-d-\alpha} dy. \end{aligned} \quad (3.2)$$

Before we estimate this last integral we need to introduce a new geometric context.

We simplify the notation by centering at 0, so that F becomes $\{z \in \mathbb{R}^d : z_d = 0\}$ and can be identified with \mathbb{R}^{d-1} . Namely, if $z \in B(0, 1)$, then we consider the circle (or line) passing through $\mathbf{1}$, $-\mathbf{1}$, and z , which intersects the hyperplane $\{z_d = 0\}$ at $z^* = (\tilde{z}^*, 0)$, a unique point in $B(0, 1)$. The situation is shown on Figure 2a. We denote by r and $b = (\tilde{b}, 0)$ the radius and the center of the circle. Namely,

$$b = -z^* \frac{1 - |z^*|^2}{2|z^*|^2}, \quad r = \frac{1 + |z^*|^2}{2|z^*|}, \quad (3.3)$$

because direct verification gives that

$$r^2 = |b - \mathbf{1}|^2 = |b - z^*|^2 = |b - z|^2 = |b + \mathbf{1}|^2.$$

Of course, $r > 1$. In particular, $r^2 = |b - \tilde{z}|^2 + z_d^2 = (|b| + |\tilde{z}|)^2 + z_d^2$, and so $|\tilde{z}| = \sqrt{r^2 - z_d^2} - |b|$. We see that

$$|z^*| - |\tilde{z}| = |z^*| + |b| - \sqrt{r^2 - z_d^2} = r - \sqrt{r^2 - z_d^2} = \frac{z_d^2}{r + \sqrt{r^2 - z_d^2}}.$$

By (3.3), $|z^*|/\sqrt{2} \leq 1/r \leq \sqrt{2}|z^*|$. Therefore,

$$z_d^2|z^*|/(2\sqrt{2}) \leq |z^*| - |\tilde{z}| \leq \sqrt{2}z_d^2|z^*|, \quad z \in B(0,1). \quad (3.4)$$

We note that $|z^*|^2 - |\tilde{z}|^2 = (|z^*| + |\tilde{z}|)(|z^*| - |\tilde{z}|)$, therefore

$$z_d^2|z^*|^2/(2\sqrt{2}) \leq |z^*|^2 - |\tilde{z}|^2 \leq 2\sqrt{2}z_d^2|z^*|^2, \quad z \in B(0,1). \quad (3.5)$$

Lemma 3.1. *For $z \in L_\varepsilon$ we have*

$$\begin{aligned} (\varepsilon^2 - |\tilde{z}|^2)^{\alpha/2} - (\varepsilon^2 - |z^*|^2)^{\alpha/2} &= \varepsilon^\alpha \left([1 - (|\tilde{z}|/\varepsilon)^2]^{\alpha/2} - [1 - (|z^*|/\varepsilon)^2]^{\alpha/2} \right) \\ &\leq (|z^*|^2 - |\tilde{z}|^2) \left([\varepsilon^2 - |z^*|^2]^{\alpha/2-1} \wedge [|z^*|^2 - |\tilde{z}|^2]^{\alpha/2-1} \right). \end{aligned}$$

Proof. Recall that $0 < \alpha < 2$. We now analyze Hölder continuity of s_ε . Note that

$$t^{\alpha/2} = \frac{\alpha}{2} \int_0^t r^{\alpha/2-1} dr, \quad t \geq 0.$$

Therefore, if $0 < t < s$, then

$$\begin{aligned} t^{\alpha/2} - s^{\alpha/2} &= s^{\alpha/2} \frac{\alpha}{2} \int_1^{t/s} r^{\alpha/2-1} dr \leq s^{\alpha/2} \frac{\alpha}{2} \int_1^{t/s} 1 dr = \frac{\alpha}{2} (t - s) s^{\alpha/2-1}, \\ t^{\alpha/2} - s^{\alpha/2} &= \frac{\alpha}{2} \int_s^t r^{\alpha/2-1} dr \leq \frac{\alpha}{2} \int_0^{t-s} r^{\alpha/2-1} dr = (t - s)^{\alpha/2}, \end{aligned}$$

and so

$$t^{\alpha/2} - s^{\alpha/2} \leq (t - s) \left(s^{\alpha/2-1} \wedge (t - s)^{\alpha/2-1} \right).$$

□

We remark that the above is sharp, meaning that a proportional lower bound holds, too. Indeed, let $f(x) = x^{\alpha/2} - 1$ if $x \geq 1$. If $x \in [1, 2]$, then $f'(x) = (\alpha/2)x^{\alpha/2-1} \geq \alpha/4$, and so $f(x) \geq \alpha(x - 1)/4$. Note that $2^{-\alpha/2} - 1 \geq \alpha/4$. If $x \geq 2$, then $f(x) \geq x^{\alpha/2} - (x/2)^{\alpha/2} = x^{\alpha/2}(1 - 2^{-\alpha/2}) \geq (x - 1)^{\alpha/2} 2^{-\alpha/2} (2^{\alpha/2} - 1) \geq (x - 1)^{\alpha/2} \alpha/8$. Therefore,

$$\begin{aligned} t^{\alpha/2} - s^{\alpha/2} &= s^{\alpha/2} \left[\left(\frac{t}{s} \right)^{\alpha/2} - 1 \right] \geq \frac{\alpha}{8} s^{\alpha/2} \left[\left(\frac{t}{s} - 1 \right) \wedge \left(\frac{t}{s} - 1 \right)^{\alpha/2} \right] \\ &= \frac{\alpha}{8} (t - s) \left[s^{\alpha/2-1} \wedge (t - s)^{\alpha/2-1} \right]. \end{aligned}$$

When L_ε is centered at 0, the integral (3.2) becomes

$$\int_{\Pi_\varepsilon} h(z + \mathbf{1}) [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz. \quad (3.6)$$

Recall that for $z \in \mathbb{R}^d$ we consider the decomposition $z = (\tilde{z}, z_d)$. We split this integral by considering subsets of Π_ε . Let $U = \Pi_\varepsilon \cap (\{z_d \leq -3/2\} \cup \{z_d \geq 1/2\})$, $V = \Pi_\varepsilon \cap \{0 \leq z_d \leq 1/2\}$, $V_s = \Pi_\varepsilon \cap \{-1/2 \leq z_d \leq 0\}$ and $W = \Pi_\varepsilon \cap \{-3/2 \leq z_d \leq -1/2\}$. Observe that for $z \in U$ we have $|x - z| \asymp z_d$, the function $h(z + \mathbf{1})$ is bounded and

$|s_\varepsilon^*(z) - s_\varepsilon(z)| \leq C\varepsilon^\alpha$ so that the integral over U is clearly $O(\varepsilon^\alpha)$. On W the function h is integrable and the rest of the integrand is bounded by $C\varepsilon^\alpha$. This leaves us with the hard part, i.e. the integral over $V \cup V_1$. Observe that on the latter set the function $h(z + \mathbf{1}) = |z + \mathbf{1}|^{\alpha-d}$ is bounded. Hence, it is enough to estimate

$$\int_V [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz$$

(by symmetry, the integral over V_1 enjoys the same upper bound). For some $z \in V$, the point z^* is outside of Π_ε . Still,

$$|z^*| = |z^*| - |\tilde{z}| + |\tilde{z}| \leq \sqrt{2}z_d^2|z^*| + \varepsilon \leq |z^*|/2 + \varepsilon.$$

Hence $|z^*| \leq 2\varepsilon$. Therefore, by (3.5)

$$|z^*|^2 - |\tilde{z}|^2 \leq \sqrt{2}z_d^2|z^*|^2 \leq 4z_d^2\varepsilon^2. \quad (3.7)$$

Lemma 3.2. *For any $z \in V$, we have*

$$s_\varepsilon(z) - s_\varepsilon^*(z) \leq c\varepsilon^\alpha z_d^\alpha.$$

Proof. On $V \setminus L_\varepsilon$, we have $|z^*| > \varepsilon$. Hence by (3.7)

$$s_\varepsilon(z) - s_\varepsilon^*(z) = s_\varepsilon(z) = C(\varepsilon^2 - |\tilde{z}|^2)^{\alpha/2} \leq C(|z^*|^2 - |\tilde{z}|^2)^{\alpha/2} \leq C\varepsilon^\alpha z_d^\alpha$$

On $V \cap L_\varepsilon$, Lemma 3.1 and (3.7) imply

$$s_\varepsilon(z) - s_\varepsilon^*(z) \leq C(|z^*|^2 - |\tilde{z}|^2)^{\alpha/2} \leq C\varepsilon^\alpha z_d^\alpha.$$

This ends the proof. \square

The following lemma is standard, for a detailed proof see e.g. Lemma 5.1 in [20].

Lemma 3.3. *For $x \in \mathbb{R}^d$ we let $\delta_{\mathbb{S}^{d-1}}(x) = \inf\{|y - x| : y \in \mathbb{S}^{d-1}\}$. If $0 < \rho < 1$, then there is a constant C_ρ (depending on ρ) such that*

$$\int_{\mathbb{S}^{d-1}} |x - y|^{-d+\rho} d\sigma(y) \leq C_\rho \delta_{\mathbb{S}^{d-1}}^{\rho-1}(x), \quad x \in \mathbb{R}^d. \quad (3.8)$$

In \mathbb{R}^{d-1} , consider coordinates $\tilde{z} = (r, \theta)$, where $\theta \in \mathbb{S}^{d-2}$ and $r = |\tilde{z}|$. Let $S(0, R) = \{\tilde{z} \in \mathbb{R}^{d-1} : |\tilde{z}| = R\} = R\mathbb{S}^{d-2}$ (cf. Figure 2b). Let $\tilde{\sigma}$ denote the $(d-2)$ -dimensional Hausdorff measure.

Corollary 3.4. *Let $\tilde{x} \in \Pi_\varepsilon \cap F$. For $\rho < 1$ we have*

$$\int_{S(0,r)} |\tilde{x} - \tilde{y}|^{-d+1+\rho} d\tilde{\sigma}(\tilde{y}) \leq C_\rho |r - |\tilde{x}||^{-1+\rho}, \quad r > 0.$$

Proof. This follows from a simple change of variable in (3.8). \square

Now, set $\delta = (2 - \alpha)/8$ and define $D = \{z \in V : 8\varepsilon z_d^{2(1-\delta)} \leq \varepsilon - |\tilde{z}|\}$. Assume $z \in D$. Since $|z_d| \leq 1/2$, the second bound from (3.4) yields $|z^*| \leq 2|\tilde{z}| \leq 2\varepsilon$. This can be improved to $|z^*| \leq \varepsilon$, because if $|z^*| > \varepsilon$, then by (3.4)

$$8\varepsilon z_d^{2-2\delta} \leq \varepsilon - |\tilde{z}| \leq |z^*| - |\tilde{z}| \leq \sqrt{2}z_d^2|z^*| \leq 2\sqrt{2}z_d^2\varepsilon,$$

giving a contradiction with $z_d \leq 1/2$. Hence, in particular, $D \subset L_\varepsilon$. Furthermore, by (3.4) and the definition of D ,

$$\begin{aligned} \varepsilon - |z^*| &= (\varepsilon - |\tilde{z}|) - (|z^*| - |\tilde{z}|) \geq (\varepsilon - |\tilde{z}|)/2 + (\varepsilon - |\tilde{z}|)/2 - \sqrt{2}\varepsilon z_d^2 \geq \\ &\geq (\varepsilon - |\tilde{z}|)/2. \end{aligned}$$

Using the first bound from Lemma 3.1, (3.5) and the above inequality,

$$\begin{aligned} &\int_D [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz \\ &\leq C \int_D (|z^*|^2 - |\tilde{z}|^2) (\varepsilon^2 - |z^*|^2)^{\alpha/2-1} |x - z|^{-d-\alpha} dz \\ &\leq C\varepsilon^{1+\alpha/2} \int_D z_d^2 (\varepsilon - |\tilde{z}|)^{\alpha/2-1} |x - z|^{-d-\alpha} dz \\ &\leq C\varepsilon^\alpha \int_V z_d^{\alpha+(2-\alpha)\delta} |x - z|^{-d-\alpha} dz \\ &\leq C\varepsilon^\alpha \int_V |x - z|^{-d+(2-\alpha)\delta} dz < C\varepsilon^\alpha. \end{aligned}$$

Hence we need to consider integral over $V \setminus D$. For a fixed $x \in F \cap L_\varepsilon$ let $A = \{z \in V : ||\tilde{x}| - |\tilde{z}||^{1-\delta} \leq z_d\}$.

Using Lemma 3.2 and the inequality $|x - z| \geq z_d$ we get

$$\begin{aligned} \int_A [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz &\leq C\varepsilon^\alpha \int_A z_d^\alpha |\tilde{x} - \tilde{z}|^{-d+3/2} |x - z|^{-3/2-\alpha} dz \\ &\leq C\varepsilon^\alpha \int_A z_d^\alpha z_d^{-3/2-\alpha} |\tilde{x} - \tilde{z}|^{-d+3/2} dz \end{aligned}$$

In cylindrical coordinates, this can be rewritten as

$$C\varepsilon^\alpha \int_0^\varepsilon \left(\int_{S(0,r)} |\tilde{x} - \tilde{z}|^{-(d-1)+1/2} d\tilde{\sigma}(\tilde{z}) \int_{z_d \geq ||\tilde{x}| - r|^{1-\delta}} z_d^{-3/2} dz_d \right) dr.$$

By Corollary 3.4, this is bounded by

$$C\varepsilon^\alpha \int_0^\varepsilon ||\tilde{x}| - r|^{-1/2} ||\tilde{x}| - r|^{-(1-\delta)/2} dr \leq C\varepsilon^\alpha \int_0^1 ||\tilde{x}| - r|^{-1+\delta/2} dr < C\varepsilon^\alpha.$$

Now let $B = \{z \in V : z_d \leq |x - z|^{1+\delta}\}$. Lemma 3.2 gives

$$\begin{aligned} &\int_B [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz \\ &\leq C\varepsilon^\alpha \int_B z_d^\alpha |x - z|^{-d-\alpha} dz \\ &\leq C\varepsilon^\alpha \int_{B(x,1)} |x - z|^{\alpha(1+\delta)} |x - z|^{-d-\alpha} dz \leq C\varepsilon^\alpha. \end{aligned}$$

We still need an estimate on $E = V \cap (A \cup B \cup D)^c$, that is

$$E = \{z \in V : \varepsilon - |\tilde{z}| < 8\varepsilon z_d^{2-2\delta}, |x - z|^{1+\delta} < z_d < ||\tilde{x}| - |\tilde{z}||^{1-\delta}\}. \quad (3.9)$$

First, we establish some geometric properties for $z \in E$. Suppose $|\tilde{z}| < |\tilde{x}|$. Then

$$\varepsilon - |\tilde{z}| \leq 8\varepsilon z_d^{2-2\delta} \leq 8\varepsilon(|\tilde{x}| - |\tilde{z}|)^{2(1-\delta)^2} \leq 8\varepsilon(\varepsilon - |\tilde{z}|)^{2(1-\delta)^2}.$$

Since $\varepsilon < 1/8$, $\varepsilon - |\tilde{z}| \leq \varepsilon < 1$ and $\delta < 1/4 < 1 - 1/\sqrt{2}$, we get a contradiction. Therefore

$$|\tilde{x}| \leq |\tilde{z}| \leq \varepsilon \text{ on } E.$$

Let $a = \varepsilon - |\tilde{x}|$ and $e = |\tilde{z}| - |\tilde{x}| \geq 0$. Note that $e \leq |x - z| \leq z_d^{1/(1+\delta)} \leq z_d^{1-\delta}$. By assumption, we have $\delta < 1/3$. Since $z_d^2 + |\tilde{x} - \tilde{z}|^2 = |x - z|^2$ and $e \leq |\tilde{x} - \tilde{z}|$, we get $e^2 + z_d^2 \leq |x - z|^2$. Hence

$$\begin{aligned} |x - z|^2 - a^2 &\geq e^2 + z_d^2 - (\varepsilon - |\tilde{z}| + e)^2 \\ &= z_d^2 - (\varepsilon - |\tilde{z}|)^2 - 2(\varepsilon - |\tilde{z}|)e \\ &\geq z_d^2 - 64\varepsilon^2 z_d^{4-4\delta} - 16\varepsilon z_d^{2-2\delta} h^{1-\delta} \\ &= z_d^2 \left(1 - 64\varepsilon^2 z_d^{2-4\delta} - 16\varepsilon z_d^{1-3\delta}\right) \\ &\geq z_d^2 (1 - 64\varepsilon^2 - 16\varepsilon). \end{aligned}$$

The expression in the parenthesis is positive for $\varepsilon < 1/20$, giving $|x - z| \geq a$. Note also that

$$\begin{aligned} z_d &\geq |x - z|^{1+\delta} \geq a^{1+\delta} \\ z_d &\leq (|\tilde{z}| - |\tilde{x}|)^{1-\delta} = [(\varepsilon - |\tilde{x}|) - (\varepsilon - |\tilde{z}|)]^{1-\delta} \leq a^{1-\delta}. \end{aligned}$$

Since E is not a subset of L_ε , [Lemma 3.1](#) is not applicable. We have the following estimate instead

$$s_\varepsilon(z) - s_\varepsilon^*(z) \leq s_\varepsilon(z) \leq C\varepsilon^{\alpha/2}(\varepsilon - |\tilde{z}|)(\varepsilon - |\tilde{z}|)^{\alpha/2-1} \leq C\varepsilon^{\alpha/2}\varepsilon z_d^{2-2\delta}(\varepsilon - |\tilde{z}|)^{\alpha/2-1}.$$

Note also that $|x - z|^{-d-\alpha} \leq |x - z|^{-d} z_d^{-\alpha} \leq |x - z|^{-d+1+\alpha/2} z_d^{-1-\alpha/2-\alpha}$. Since $-d + 1 + \alpha/2 < 0$, we get

$$|x - z|^{-d-\alpha} \leq |\tilde{x} - \tilde{z}|^{-d+1+\alpha/2} z_d^{-1-\alpha/2-\alpha}.$$

By asumption, $\delta + \alpha < 2$. Using cylindrical coordinates, we obtain,

$$\begin{aligned}
& \int_E [s_\varepsilon(z) - s_\varepsilon^*(z)] |x - z|^{-d-\alpha} dz \\
& \leq C\varepsilon^{\alpha/2+1} \int_{a^{1+\delta}}^{a^{1-\delta}} z_d^{1-\alpha/2-\alpha-2\delta} dz_d \int_{\varepsilon \geq |\tilde{z}| \geq |\tilde{x}|} |\tilde{x} - \tilde{z}|^{-d+1+\alpha/2} (\varepsilon - |\tilde{z}|)^{\alpha/2-1} d\tilde{z} \\
& \leq C\varepsilon^{\alpha/2+1} a^{-(1+\delta)(\alpha+2\delta)/2} \int_0^{a^{1-\delta}} z_d^{1-\alpha/2-(\alpha+2\delta)/2} dz_d \times \\
& \quad \times \int_{|\tilde{x}|}^\varepsilon (\varepsilon - r)^{\alpha/2-1} \int_{S(0,r)} |\tilde{x} - \tilde{z}|^{-(d-1)+\alpha/2} d\sigma(\tilde{z}) dr \\
& \leq C\varepsilon^{\alpha/2+1} a^{-(1+\delta)(\alpha+2\delta)/2} a^{(1-\delta)(2-\alpha/2-(\alpha+2\delta)/2)} \int_{|\tilde{x}|}^\varepsilon (\varepsilon - r)^{\alpha/2-1} (r - |\tilde{x}|)^{\alpha/2-1} dr \\
& = C\varepsilon^{\alpha/2+1} a^{-\alpha-2\delta+(1-\delta)(2-\alpha/2)} (\varepsilon - |\tilde{x}|)^{\alpha-1} \int_0^1 r^{\alpha/2-1} (1-r)^{\alpha/2-1} dr \\
& = C\varepsilon^{\alpha/2+1} a^{-\alpha-2\delta+(1-\delta)(2-\alpha/2)+\alpha-1} = C\varepsilon^{\alpha/2+1} a^{1-\alpha/2-\delta(4-\alpha/2)}.
\end{aligned}$$

Since $\delta = (2 - \alpha)/8$, we get a in a positive power. Consequently, the integral decays slightly faster than ε^α . This completes the proof of [Lemma 2.4](#).

3.3 Proof of [Lemma 2.7](#)

Let $t \in [-1, 1)$ and $r \in [0, 1]$. By [\(2.8\)](#),

$$u_\lambda(t) - u_0(t) = \int_0^1 (1 - r^\lambda) (r^{\alpha-\lambda-1} - r^{d-1}) (r^2 - 2rt + 1)^{-(d+\alpha)/2} dr. \quad (3.10)$$

Let $g(r) = r^2 - 2rt + 1 = (r - t)^2 + 1 - t^2$ and $f(r) = g(r)^{-(d+\alpha)/2}$. To estimate f we observe that the extrema of $g(r)$ on $[0, 1]$ are either $g(0) = 1$, or $g(1) = 2(1 - t)$ or $g(t) = 1 - t^2$. We first prove the lower bound in the statement of the lemma. Note that $f(0) = 1$. We have $f'(r) = -(d + \alpha)(r - t)g(r)^{-(d+\alpha)/2-1}$, hence $f'(r) \geq -2(d + \alpha)$ if $-1 \leq t < 0$, and $f'(r) \geq -(d + \alpha)(1 - t^2)^{-(d+\alpha)/2-1} \geq -(d + \alpha)(4/3)^{(d+\alpha)/2+1}$ if $0 \leq t < 1/2$. If $1/2 \leq t < 1$, then $g(r) = r(r - 2t) + 1 \leq 1$ and $f(r) \geq 1$. Summarizing, in each case we have $f(r) \geq 1 - cr$. Therefore,

$$\begin{aligned}
u_\lambda(t) - u_0(t) & \geq \int_0^1 (1 - r^\lambda) (r^{\alpha-\lambda-1} - r^{d-1}) (1 - cr) dr \\
& = \frac{1}{\alpha - \lambda} - \frac{1}{d} - \frac{1}{\alpha} + \frac{1}{d + \lambda} - c \left(\frac{1}{\alpha - \lambda + 1} - \frac{1}{d + 1} - \frac{1}{\alpha + 1} + \frac{1}{d + \lambda + 1} \right),
\end{aligned} \quad (3.11)$$

and the lower bound in the statement of the lemma follows. We now prove the upper bound. If $t \leq 3/4$, then $g(r) \geq 1 - (3/4)^2$, hence $|f'(r)| \leq c$ and $f(r) \leq 1 + cr$. This and [\(3.10\)](#) yield $u_\lambda(t) - u_0(t) \leq \frac{1}{\alpha - \lambda} + C$, cf. [\(3.11\)](#).

We denote $s = \sqrt{2(1 - t)}$, so that $s \in (0, 2]$. If $\theta, \eta \in \mathbb{S}^{d-1}$, $t = \theta \cdot \eta$ and γ is the angle between θ and η , then $t = \cos \gamma$ and

$$s = 2 \sin(\gamma/2) = |\theta - \eta|. \quad (3.12)$$

We need to consider $t > 3/4$, or $s^2 < 1/2$. Let $x \in (0, 1/2)$. By (3.10),

$$\begin{aligned} u_\lambda(t) - u_0(t) &= \left(\int_0^x + \int_x^1 \right) (1 - r^\lambda) (r^{\alpha-\lambda-1} - r^{d-1}) ((r-1)^2 + rs^2)^{-\frac{d+\alpha}{2}} dr \\ &= \text{I} + \text{II}. \end{aligned}$$

We have

$$\begin{aligned} \text{I} &\leq \int_0^x r^{\alpha-\lambda-1} (1-r)^{-d-\alpha} dr \leq (1-x)^{-d-\alpha} \int_0^1 r^{\alpha-\lambda-1} dr \\ &= \frac{(1-x)^{-d-\alpha}}{\alpha-\lambda} \leq \frac{1+Cx}{\alpha-\lambda}. \end{aligned}$$

To estimate II, we denote $f_a^s(v) = (1+vs)^a$ and

$$\Delta f_a^s(v) = \frac{f_a^0(v) - f_a^s(v)}{s} = \frac{1 - f_a^s(v)}{s},$$

where $a > 0$, $s > 0$ and $-1/s \leq v \leq 0$. For $0 < y < 1$ and $a > 0$ we have

$$(1-y)^a \geq (1-y)^{a \vee 1} \geq 1 - (a \vee 1)y.$$

Therefore $(1 - (1-y)^a)/y \leq a \vee 1$. Putting $y = |v|s$ we get

$$\Delta f_a^s(v) \leq (a \vee 1)|v|. \quad (3.13)$$

Substituting $r-1 = vs$ in the integral defining II, we get

$$\begin{aligned} \text{II} &\leq x^{\alpha-\lambda-1} \int_x^1 (1-r^\lambda) (1-r^{d-\alpha+\lambda}) ((r-1)^2 + rs^2)^{-\frac{d+\alpha}{2}} dr \\ &= x^{\alpha-\lambda-1} s^{-d-\alpha+1} \int_{(x-1)/s}^0 (1-f_\lambda^s(v)) (1-f_{d-\alpha+\lambda}^s(v)) (v^2 + (1+vs))^{-\frac{d+\alpha}{2}} dv \\ &= x^{\alpha-\lambda-1} s^{-d-\alpha+3} \int_{(x-1)/s}^0 \Delta f_\lambda^s(v) \Delta f_{d-\alpha+\lambda}^s(v) (v^2 + vs + 1)^{-\frac{d+\alpha}{2}} dv. \end{aligned}$$

Note that $-1 \leq x-1 \leq vs \leq 0$. By (3.13) we have

$$\begin{aligned} \Delta f_\lambda^s(v) &\leq (\lambda \vee 1)|v| \leq 2|v|, \\ \Delta f_{d-\alpha+\lambda}^s(v) &\leq ((d-\alpha+\lambda) \vee 1)|v| \leq d|v|. \end{aligned}$$

Recall that $s^2 < 1/2$. By the above and a change of variables it follows that

$$\begin{aligned} \text{II} &\leq C s^{-d-\alpha+3} x^{\alpha-\lambda-1} \int_{(x-1)/s}^0 \frac{v^2}{(v^2 + vs + 1)^{(d+\alpha)/2}} dv \\ &\leq C s^{-d-\alpha+3} x^{\alpha-\lambda-1} \left(\int_{-s}^0 \frac{v^2}{(v^2 + 1/2)^{(d+\alpha)/2}} dv + \int_{-1/s}^{-s} \frac{(-v)\sqrt{v^2 + vs + 1}}{(v^2 + vs + 1)^{(d+\alpha)/2}} dv \right) \\ &\leq C s^{-d-\alpha+3} x^{\alpha-\lambda-1} \left(\int_{-1}^0 \frac{v^2}{(v^2 + 1/2)^{(d+\alpha)/2}} dv + \int_{-1/s}^{-s} \frac{-(2v+s)}{(v^2 + vs + 1)^{(d+\alpha-1)/2}} dv \right) \\ &\leq C s^{-d-\alpha+3} x^{\alpha-\lambda-1} \left(1 + \int_1^{1/s^2} \frac{du}{u^{(d+\alpha-1)/2}} \right) \\ &\leq C x^{\alpha-\lambda-1} (1 \vee s^{-d-\alpha+3}), \end{aligned}$$

provided $d+\alpha \neq 3$, and $\text{II} \leq -x^{\alpha-\lambda-1} \log s$ if $d+\alpha = 3$. Let $x = (\alpha-\lambda)^\delta$. Since $(\alpha-\lambda)^{\delta(\alpha-\lambda)} \leq 1$, the upper bound follows. The proof of Lemma 2.7 is complete.

3.4 Proof of Lemma 2.8

We consider transformation

$$W(y) = \frac{2}{|y|^2}y - \mathbf{1}, \quad y \in \mathbb{R}_0^d.$$

Note that $W^{-1}(y) = T\left(\frac{1}{2}(\mathbf{1} + y)\right)$, $y \neq -\mathbf{1}$. For $y = (w, 1) \in F$, we have

$$W(y) = \begin{bmatrix} 2(1 + |w|^2)^{-1}w \\ -1 + 2(1 + |w|^2)^{-1} \end{bmatrix}, \quad \frac{\partial W}{\partial w} = \begin{bmatrix} 2(1 + |w|^{-1}I_{d-1} - 4(1 + |w|^2)^{-2}ww^T \\ -4(1 + |w|^2)^{-2}w^T \end{bmatrix},$$

where $w \in \mathbb{R}^{d-1}$ is considered as a column vector, w^T is the transpose of w and I_{d-1} is the $(d-1) \times (d-1)$ identity matrix. By [22, Proposition 12.13] the surface measure on $W(F) = \mathbb{S}^{d-1} \setminus \{-\mathbf{1}\}$ is given by

$$\sigma(W(dw)) = \left[\det \left(\frac{\partial W^T}{\partial w} \frac{\partial W}{\partial w} \right) \right]^{1/2} dw. \quad (3.14)$$

We have

$$\frac{\partial W^T}{\partial w} \frac{\partial W}{\partial w} = 4(1 + |w|^2)^{-2}I_{d-1} + 16|w|^2(1 + |w|^2)^{-4}ww^T.$$

By the matrix determinant lemma (see, e.g, [15, Corollary 18.1.3]),

$$\det \left(\frac{\partial W^T}{\partial w} \frac{\partial W}{\partial w} \right) = (4(1 + |w|^2)^{-2})^{d-1} (1 + 4|w|^4(1 + |w|^2)^{-2}),$$

therefore $\sigma(W(dw)) = 2^{d-1}[1 + O(\varepsilon^2)]dw$ if $|w| < \varepsilon$. We have $W(F \cap \Pi_\varepsilon) = B_\Theta$ (see (2.14) and Figure 1). In fact $\phi(W(y)) = 2^\alpha s_\varepsilon^*(y) = 2^\alpha s_\varepsilon(y)$ for $y \in F \cap \Pi_\varepsilon$, cf. (2.19). Thus,

$$\begin{aligned} \int_{B_\Theta} \phi(\theta) \sigma(d\theta) &= 2^{d-1+\alpha} [1 + O(\varepsilon^2)] \int_{\mathbb{R}^{d-1} \cap \{|w| < \varepsilon\}} C_{d-1,\alpha} (\varepsilon^2 - |w|^2)^{\alpha/2} dw \\ &= 2^{d-1+\alpha} \omega_{d-1} C_{d-1,\alpha} [1 + O(\varepsilon^2)] \int_0^\varepsilon (\varepsilon^2 - r^2)^{\alpha/2} r^{d-2} dr \\ &= 2^{d-1+\alpha} \omega_{d-1} C_{d-1,\alpha} \varepsilon^{\alpha+d-1} [1 + O(\varepsilon^2)] \int_0^1 (1 - r^2)^{\alpha/2} r^{d-2} dr \\ &= 2^{d+\alpha-2} \omega_{d-1} C_{d-1,\alpha} B\left(1 + \frac{\alpha}{2}, \frac{d-1}{2}\right) \varepsilon^{\alpha+d-1} [1 + O(\varepsilon^2)]. \end{aligned}$$

We finish the proof of Lemma 2.8 by recalling that $\varepsilon = \Theta/2 + O(\Theta^2)$, cf. (2.14).

3.5 Proof of Lemma 2.9

Recall that $0 < \varepsilon < 1/20$ and $\Theta \leq \pi/30$, in particular $|\theta - \eta| \leq 1$ if $\theta \in B_\Theta$, cf. (3.12). By (2.10), the definition of ϕ , Lemma 2.7 and (2.14) we have

$$\begin{aligned} \mathcal{A}_{d,\alpha} \frac{1 - c(\alpha - \beta)}{\alpha - \lambda} \int_{B_\Theta} \phi(\theta) \sigma(d\theta) &\leq R_\lambda \phi(\eta) \\ &\leq \mathcal{A}_{d,\alpha} \frac{1 + C(\alpha - \lambda)^\delta}{\alpha - \lambda} \int_{B_\Theta} \phi(\theta) \sigma(d\theta) + C \frac{\Theta^\alpha}{(\alpha - \lambda)^\delta} \int_{B_\Theta} \left(1 \vee |\theta - \eta|^{-(d+\alpha-3)}\right) \sigma(d\theta). \end{aligned}$$

The lower bound in [Lemma 2.9](#) follows immediately from [Lemma 2.8](#).

To prove the upper bound we first assume that $d + \alpha \leq 3$. Then $d = 2$, $\alpha \leq 1$ and $\Theta^\alpha \leq \omega(\Theta)$. Since $\int_{B_\Theta} \sigma(d\theta) \asymp \Theta^{d-1} = \Theta$, by [Lemma 2.8](#) we obtain,

$$R_\lambda \phi(\eta) \leq \mathcal{A}_{d,\alpha} \tilde{C}_{d,\alpha} \Theta^{d-1+\alpha} [1 + O(\Theta^2)] \frac{1 + C(\alpha - \lambda)^\delta}{\alpha - \lambda} + C \frac{\Theta^{1+\alpha}}{(\alpha - \lambda)^\delta},$$

as needed. We now assume that $d + \alpha > 3$. It is not difficult to see that

$$\int_{B_\Theta} |\eta - \theta|^{-(d+\alpha-3)} \sigma(d\theta) \asymp \Theta^{2-\alpha}.$$

Consequently,

$$R_\lambda \phi(\eta) \leq \mathcal{A}_{d,\alpha} \tilde{C}_{d,\alpha} \Theta^{d-1+\alpha} [1 + O(\Theta^2)] \frac{1 + C(\alpha - \lambda)^\delta}{\alpha - \lambda} + C \frac{\Theta^2}{(\alpha - \lambda)^\delta}.$$

But $\Theta \leq \omega(\Theta)$, which yields [\(2.22\)](#) in this case, too. The proof of [Lemma 2.9](#) is complete. In fact we proved a stronger estimate for $\alpha = 1$.

3.6 Proof of [Corollary 1.2](#)

The following is a folklore connection between harmonic functions of $\Delta^{1/2}$ and Δ .

Lemma 3.5. *Let $d \in \{1, 2, \dots\}$ and*

$$P_t(x) = \frac{2}{\omega_{d+1}} \frac{t}{(|x|^2 + t^2)^{(d+1)/2}}, \quad t > 0, x \in \mathbb{R}^d.$$

If function Φ on \mathbb{R}^n is harmonic for $\Delta^{1/2}$ in a open set $E \subset \mathbb{R}^d$, and for $x \in \mathbb{R}^d$ we let

$$U(x, t) = \begin{cases} P_t * \Phi(x), & \text{if } t > 0, \\ \Phi(x), & \text{if } t = 0, \\ P_{-t} * \Phi(x), & \text{if } t < 0, \end{cases}$$

then U is harmonic for Δ in $D = \{(x, t) \in \mathbb{R}^{d+1} : t \neq 0 \text{ or } x \in E\}$.

Proof. U is well-defined because of [\(2.1\)](#). It is harmonic (for Δ in $d + 1$ variables) on $\mathbb{R}^{d+1} \setminus \{t = 0\}$ and continuous on D , cf. [\[21, Chapter III\]](#). It is well-known and easy to derive directly that $\partial U(x, t)/\partial t = \Delta^{1/2} \Phi(x)$ at $t = 0$ and $x \in E$ (hint: $\int_{\mathbb{R}^d} P_t(y) dy = 1$). Since Φ is $1/2$ -harmonic on E , the derivative equals zero at $t = 0$ and $x \in E$. It follows that $V(x, t) = \partial U(x, t)/\partial t$ is continuous in D . By the reflection principle for harmonic functions, V is harmonic in D . For $x \in D$ and $t \in \mathbb{R}$ we have $U(x, t) = U(x, 1) + \int_1^t V(x, s) ds$. Thus U is C^2 in D , and so $\Delta U = 0$ on D . \square

Let M be the Martin kernel of the cone $\Gamma_\Theta \subset \mathbb{R}^d$ for $\Delta^{1/2}$ and $d \geq 2$. The above harmonic extension of M to \mathbb{R}^{d+1} is a constant multiple of the Martin kernel (with the pole at infinity) for V and Δ . Indeed, [\[2, Corollary 1\]](#) asserts that all nonnegative harmonic functions vanishing at E are proportional, see also [\[16, Theorem 1.1\]](#). By [Theorem 1.1](#) and a change of variables, $M * P_{kt}(kx) = k^\beta M * P_t(x)$, where $k > 0$. We have $\beta = 1 - B_{d,1} \Theta^d + O(\Theta^{d+1} \log \Theta)$ and

$$B_{d,1} = \frac{1}{2\pi} \frac{d-1}{d} \frac{\Gamma(\frac{d-1}{2})^2}{\Gamma(\frac{d}{2})^2}.$$

Since $B_{2,1} = 1/4$, [Corollary 1.2](#) follows. In fact we proved a more general result.

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